SELF-ORGANIZED CRITICALITY

MICHAEL CREUTZ
Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

I review the concept of self-organized criticality, wherein dissipative systems naturally drive themselves to a critical state with important phenomena occurring over a wide range of length and time scales. Several exact results are demonstrated for the Abelian sandpile.

Self-organized criticality concerns a class of dynamical systems which naturally drive themselves to a state where interesting physics occurs on all scales[1]. The idea provides a possible "explanation" of the omnipresent multi-scale structures throughout the natural world, ranging from the fractal structure of mountains, to the power law spectra of earthquake sizes [2]. Recent applications include such diverse topics as evolution[3] and traffic flow [4]. The concept has even been invoked to explain the unpredictable nature of economic systems, i.e. why you can't beat the stock market [5].

The prototypical example is a sandpile. On slowly adding grains of sand to an empty table, a pile will grow until its slope becomes critical and avalanches spill over the sides. If the slope becomes too large, a large catastrophic avalanche is likely, and the slope will reduce. If the slope is too small, then the sand will accumulate to make the pile steeper. Ultimately one should obtain avalanches of all sizes, with the prediction for the next being impossible without actually running the experiment.

Self-organized criticality nicely compliments the concept of chaos. In the latter, dynamical systems with a few degrees of freedom, say three or more, can display highly complex behavior, including fractal structures. With self-organized criticality, we start instead with systems of many degrees of freedom, and find a few general common features. Another attraction of this topic is the ease with which computer models can be implemented and the elegance of the resulting graphics [6].

The original Bak, Tang, Wiesenfeld paper [1] presented a simple model wherein each site in a two dimensional lattice has a state specified by a positive integer z_i. This can be thought of as the amount of sand at that location, or, in another sense, as the slope of the sandpile at that point. Neither of these analogies is fully accurate, for the model has aspects of each.

The dynamics follows by setting a threshold z_T above which any given z_i is unstable. Without loss of generality, I take this threshold to be $z_T = 3$. Time now proceeds in discrete steps. In one such step each unstable site "tumbles"

or "topples," dropping by four and adding one grain to each of its four nearest neighbors. This may produce other unstable sites, and thus an avalanche can ensue for further time steps until all sites are stable. Fig. 1 shows a typical configuration on a 198 by 198 lattice after lots of random sand addition followed by relaxation. Fig. 2 shows an avalanche proceeding on this lattice.

A natural experiment consists of adding a grain of sand to a random site and measuring the number of topplings and the number of time steps for the resulting avalanche. Repeating this many times to gain statistics, the distribution of avalanche sizes and lengths displays a power law behavior, with all sizes appearing. In Ref. [7] such experiments showed that the distribution of the number of tumbling events s in an avalanche empirically scales as

$$P(s) \sim s^{-1.07} \tag{1}$$

and the number of time steps τ for avalanches scales as

$$P(\tau) \sim \tau^{-1.14} \tag{2}$$

This model has been extensively studied analytically. While as yet there is no exact calculation of these exponents, a lot is known. In particular, the critical ensemble is well characterized. I will return to these points later.

The extent to which laboratory experiments reproduce such power laws is somewhat controversial. A recent study of avalanche dynamics [8] in rice piles showed criticality with long-grain rice, but more ambiguous results followed similar experiments with short-grain rice.

Another simple model mimics forest fires and has three possible states per cell, empty, a tree, or a fire. For the updating step, any empty site can have a tree born with a small probability. At the same time, any existing fire spreads to neighboring trees leaving its own cell empty. The random growth of trees gives this rule a stochastic nature. As the system is made larger, the growth rate for the trees should decrease to just enough to keep the fires going.

If too many trees grow, one obtains a large fire reducing their density, while if there are too few trees, fires die out. On a finite system, one should light a fire somewhere to get the system started. As the system becomes larger, the growth rate for the trees can be reduced without the fire expiring. In a steady state the system has fire fronts continually passing through the system, as illustrated in Fig. 3a. Perhaps there is a moral here to be careful about extinguishing all fires in the real world, for this may enhance the possibility for a catastrophic uncontrollable fire. It is not entirely clear whether this model is actually critical. What seems to happen on large systems is that stable spiral structures form and set up a steady rotation. For a review of this and several related models, see Ref. [9].

A variation on this model has several "species" of fires. Perhaps a better metaphor is to think of different species of bunny, competing for the same slowly growing food resource. With the four cell neighborhood, a natural division into species is given by the parity of the site plus the time step. Fig. 3b. shows a state in the evolution of such a model when both species are present. This situation, however, is highly unstable, with any fluctuation favoring one species tending to grow until the competitor is eliminated. This model provides a discrete realization of the "principle of competitive exclusion" in biological systems [10]. Stability of a species requires that it occupy its own niche and not compete for exactly the same resources as another.

Very little rigorous is known about general self-organized critical systems. However, in a series of papers, Deepak Dhar and co-workers have shown that the sandpile model has some rather remarkable mathematical properties[11,12,13,14]. In particular, the critical ensemble of the system has been well characterized in terms of an Abelian group. In the following I will generally follow the discussion given in Refs. [15,2].

Dhar[11] introduced the useful toppling matrix $\Delta_{i,j}$ with integer elements representing the change in the height, z at site i resulting from a toppling at site j. More precisely, under a toppling at site j, the height at any site i becomes $z_i - \Delta_{i,j}$. For the simple two dimensional sand model the toppling matrix is thus

$$\begin{array}{lll} \Delta_{i,j} & = 4 & i = j \\ \Delta_{i,j} & = -1 & i,j \text{ nearest neighbors} \\ \Delta_{i,j} & = 0 & \text{otherwise.} \end{array} \tag{3}$$

For this discussion there is little special to the specific lattice geometry; indeed, the following results easily generalize to other lattices and dimensions. The analysis requires only that under a toppling of a single site i, that site has its slope decreased ($\Delta_{i,i} > 0$), the slope at any other site is either increased or unchanged ($\Delta_{i,j} \leq 0, j \neq i$), the total amount of sand in the system does not increase ($\sum_j \Delta_{i,j} \geq 0$), and, finally, that each site be connected through toppling events to some location where sand can be lost, such as at a boundary.

For the specific case in Eq. 3, the sum of slopes over all sites is conserved whenever a site away from the lattice edge undergoes a toppling. Only at the lattice boundaries can sand be lost. Thus the details of this model depend crucially on the boundaries, which we take to be open. A toppling at an edge loses one grain of sand and at a corner loses two.

The actual value of the maximum stable height z_T is unimportant to the dynamics. This can be changed by simply adding constants to all the z_i. Thus without loss of generality I consider $z_T = 3$. With this convention, if all z_i are initially non-negative they will remain so, and I thus restrict myself to states

C belonging to that set. The states where all z_i are positive and less than 4 are called stable; a state that has any z_i larger than or equal to 4 is called unstable. One conceptually useful configuration is the minimally stable state C^* which has all the heights at the critical value z_T. By construction, any addition of sand to C^* will give an unstable state leading to a large avalanche.

I now formally define various operators acting on the states C. First, the "sand addition" operator α_i acting on any C yields the state $\alpha_i C$ where $z_i = z_i + 1$ and all other z are unchanged. Next, the toppling operator t_i transforms C into the state with heights z'_j where $z'_j = z_j - \Delta_{i,j}$. The operator U which updates the lattice one time step is now simply the product of t_i over all sites where the slope is unstable,

$$UC = \prod_i t_i^{p_i} C \tag{4}$$

where $p_i = 1$ if $z_i \geq 4$; 0 otherwise. Using U repeatedly gives the relaxation operator R. Applied to any state C this corresponds to repeating U until no more z_i change. Neither U nor R have any effect on stable states. Finally, I define the avalanche operators a_i describing the action of adding a grain of sand to site i followed by relaxation

$$a_i C = R \alpha_i C. \tag{5}$$

At this point it is not entirely clear that the operator R exists; in particular, it might be that the updating procedure enters a non-trivial cycle consisting of a never ending avalanche. This, however, is impossible as can be shown from the fact that sand spreads during an avalanche.

With an edge-less system, such as under periodic boundaries, no sand would be lost and thus cycles are expected and easily observed. These models might be called "Escher models" after the artist constructing drawings of water flowing perpetually downhill and yet circulating in the system. While little is known about the dynamics of this variation on the sandpile model, some studies have been done under the nomenclature of "chip-firing games" [16]. A recent paper [17] has argued that this lossless sandpile model on an appropriate lattice is capable of universal computation.

I now introduce the concept of recursive states. This set, denoted $\mathcal{R}$, includes those stable states which can be reached from any stable state by some addition of sand followed by relaxation. This set is not empty because it contains at least the minimally stable state C^*. Indeed, that state can be obtained from any other by carefully adding just enough sand to each site to make z_i equal to three. Thus, one might alternatively define $\mathcal{R}$ as the set of

states which can be obtained from C^* by acting with some product of the operators a_i.

It is easily shown that there exist non-recursive, transient states; for instance, no recursive state can have two adjacent heights both being zero. If you try to tumble one site to zero height, then it drops a grain of sand on its neighbors. If you then tumble a neighbor to zero, it dumps a grain back on the original site. One can also show that the self-organized critical ensemble, reached under random addition of sand to the system, has equal probability for each state in the recursive set. This is a consequence of the Abelian nature of this system discussed below.

The crucial results of Refs. [11,12,13,14] are that the operators a_i acting on stable states commute. Furthermore, when restricted to recursive states these operators are invertable. Thus they generate an Abelian group. An intuitive argument that sand addition may be commutative uses an analogy with combining many digit numbers under long addition. The tumbling operation is much like carrying, except rather than to the next digit the overflow spreads to several neighbors. As addition is known to be Abelian, despite the confusing elementary-school rules, I might expect the sandpile addition rule also to be.

These results have several amusing consequences. One is a determination of the number of states in the recursive set. Without going into details, the result is the absolute value of the determinant of the toppling matrix Δ. For large lattices this determinant can be found easily by Fourier transform. In particular, whereas there are $4N$ stable states, there are only

$$\exp\left(N\int_{(-\pi,-\pi)}^{(\pi,\pi)}\frac{d^2q}{(2\pi)^2}\ln(4-2q_x-2q_y)\right)\simeq(3.2102\ldots)^N \tag{6}$$

recursive states. Thus starting from an arbitrary state and adding sand, the system “self-organizes” into an exponentially small subset of states forming the attractor of the dynamics.

Following Ref. [15], I now stack sand piles on top of one another. Given stable configurations C and C' with configurations z_i and z_i', I define the state $C\oplus C'$ to be that obtained by relaxing the configuration with heights z_i+z_i'. Clearly, if either C or C' are recursive states, so is $C\oplus C'$.

Under $\oplus$ the recursive states form an Abelian group isomorphic to the algebra generated by the a_i. The addition of a state C with heights z_i is equivalent to operating with a product of a_i raised to z_i, that is

$$B\oplus C=\left(\prod a_i^{z_i}\right)B. \tag{7}$$

The operation $\oplus$ is associative and Abelian because the operators a_i are.

Since any element of a group raised to the order of the group gives the identity, it follows that $a_i^{|\Delta|} = E$. This implies the simple formula $a_i^{-1} = a_i^{|\Delta|-1}$. The analog of this for the states is the existence of an inverse state, $-C$

$$-C = (|\Delta| - 1) \otimes C. \tag{8}$$

Here, $n \otimes C$ means adding n copies of C and relaxing. The state $-C$ has the property that for any state $B \oplus C \oplus (-C) = B$.

The state $I = C \oplus (-C)$ represents the identity and has the property $I \oplus B = B$ for every recursive state B. The state which is isomorphic to the operator a_i is simply $a_i I$. The identity state provides a simple way to check if a state, obtained for instance by a computer simulation, has reached the attractor, i.e. if a given state is a recursive state: A stable state is in $\mathcal{R}$ if and only if $C \oplus I = C$.

The identity state can be constructed by taking any recursive state, say C^* and repeatedly adding it to itself to use $|\Delta| \otimes C = I$. However, on any but the smallest lattices, $|\Delta|$ is a very large integer. A more economical scheme is given in Ref. [15]. Fig. 4 shows the identity state on a 198 by 198 lattice. Note the fractal structure, with features on many length scales.

Majumdar and Dhar [14] have constructed a simple "burning" algorithm to determine if a state belongs to the recursive set. For a given configuration, first add one particle to each of the edge sites and two particles to the corners. This corresponds to a large source of sand just outside the boundaries, which then tumbles one step onto the system. Then return to open boundaries and update according to the usual rules. If and only if the original state is recursive, this will generate an avalanche under which each site of the system tumbles exactly once. Also, the final state after the avalanche will be identical to the original. However, if the state is not recursive, some untumbled sites will remain. Fig. 5 shows such a process underway on the configuration of Fig. 1. Here sites which have already burned are shown in cyan, while the remaining sites in the center have not yet tumbled. The small number of sites shown in orange are the still active sites, which eventually burn the entire remaining lattice.

The burning algorithm provides a simple way to prove that the avalanche regions are simply connected once one is in the critical state. In a burning process, any sub-lattice of the original will have all of its sites tumbled onto from outside. This is the condition for starting a burning on the sub-lattice. Thus, if a configuration is in the critical ensemble for the whole lattice, then any extracted piece of this configuration on a subset of the original lattice is also in the critical ensemble of the extracted part. Now suppose that one constructs an avalanche with any initial addition to a state from the critical ensemble.

In any subregion enclosed by this avalanche, sand will fall from the tumbling sites on its outside. Since the sub-lattice is itself in its own critical ensemble, this must induce an avalanche which, by the burning algorithm, will tumble all enclosed sites. Thus any avalanche on a state from the critical ensemble cannot leave untumbled any sites in a region isolated from the boundary, i.e. an untumbled island. This result that avalanches must be simply connected does not follow for states outside the recursive set, as can be easily demonstrated by considering a sandpile with a hole of empty sites in the middle.

To conclude, simple cellular automaton models provide a rich area for the study of complex phenomena, and in particular for systems which self organize with physics at many scales. I have only touched on a few issues here, leaving out many related topics such as lattice gasses, driven interfaces in random media, growth processes, and evolution. As the ease of programming and the speed of modern computers continue to rush forward, so will the fascination with such systems.

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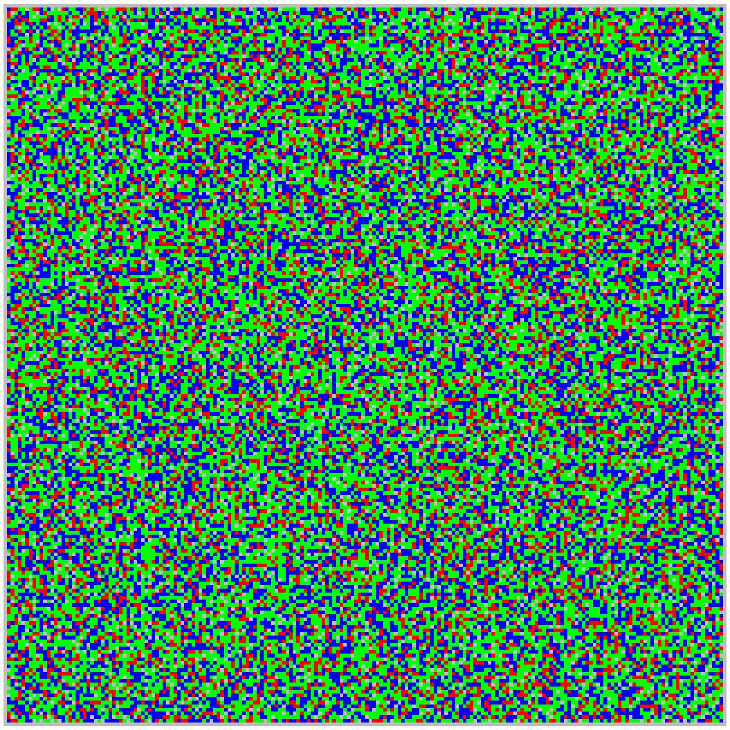

Figure 1: The sandpile model in the final stable state after adding lots of sand to random places. The lattice is 198 cells by 198 cells. The color code is grey, red, blue, and green for heights 0,1,2, and 3, respectively. Despite the lack of obvious patterns, subtle correlations are present; for example no two adjacent sites have height zero.

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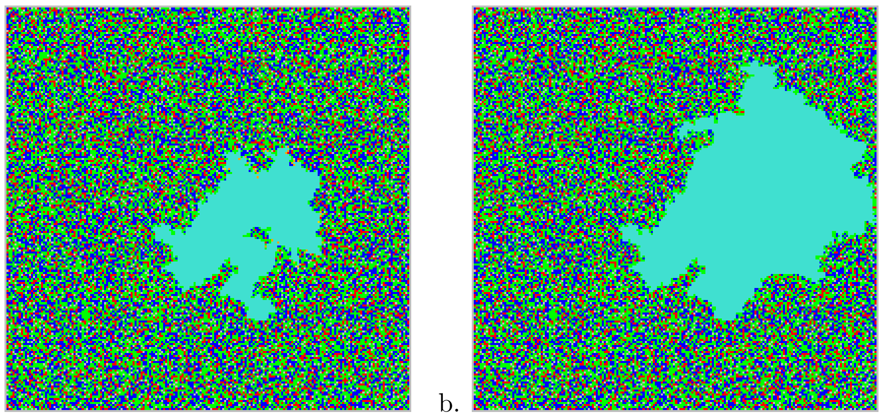
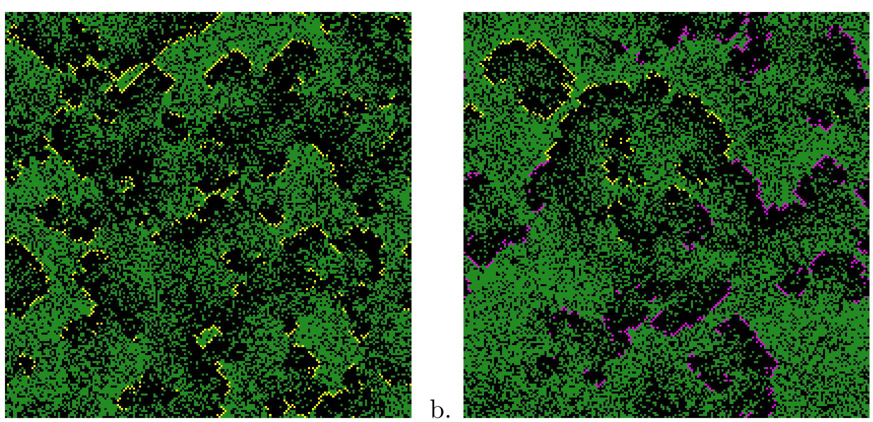


a. b.

Figure 2: An avalanche obtained by adding a small amount of sand to the configuration in Fig. 1. Stable sites which have tumbled during the avalanche are distinguished by being colored light blue. The still active sites on the left image are colored yellowish brown. The image on the right is the final state after the avalanche has ended. Note that the final avalanche region is simply connected. This is a general result proven in the text.



a. b.

Figure 3: On the left is a snapshot of the forest fire model on a 198 by 198 lattice. Trees are continuously burning at a slow rate, while fires burn them down and spread to nearest neighbor trees. Here the four cell neighborhood is used. On the right is a variation where two species of bunnies are competing to eat a common grass. The yellow and purple colors here distinguish the parity of the site plus time step. Eventually one of the two species dominates and the other dies out.

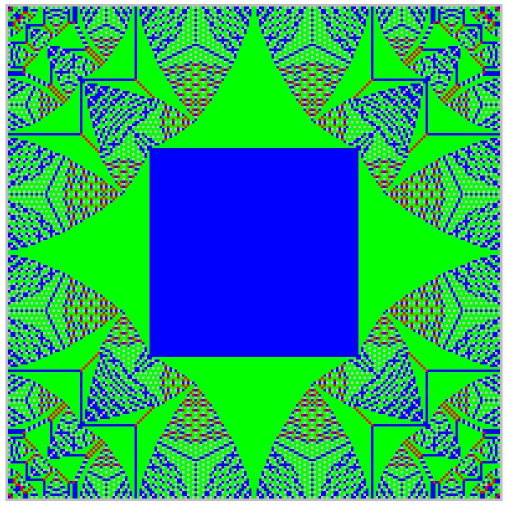


Figure 4: The identity state for the sandpile model on a 198 by 198 lattice. The color code is grey, red, blue, and green for heights 0,1,2, and 3, respectively.

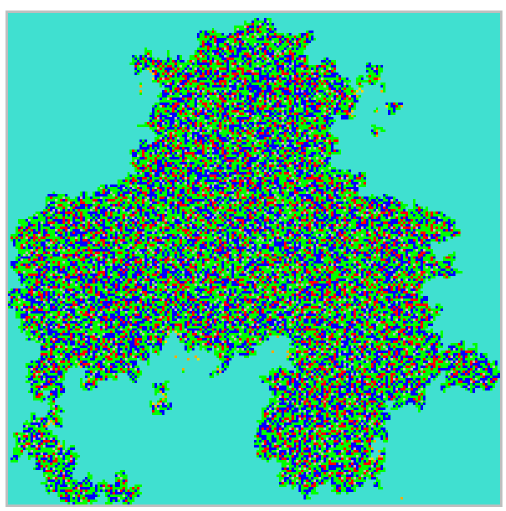


Figure 5: The burning algorithm being applied to the state in Fig. 1. Burnt sites are cyan, burning sites are orange, and the remaining sites are colored as previously. This avalanche eventually tumbles every site exactly once.